Application of Hirota's Direct Method to Nonlinear Partial Differential Equations: Bilinear Form and Soliton Solutions

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Abstract

The Hirota method to get the soliton solutions for a nonlinear partial differential equation is the most efficient direct technique researchers use worldwide. This article reviews and explores Hirota's direct technique on the KdV equation, which Hirota initially used to clarify his method. This method uses the perturbation technique to get the mathematical formulation for the N-soliton solution. We use the perturbation to the KdV equation to get one-soliton, two-soliton, and three-soliton solutions and the generalized N-soliton solution. We show the bilinear form for the selected KdV equation and the other equations used in this work. Also, we investigate the concerned method by illustrating three well-known equations, the Kadomtsev-Petviashvili (KP) equation, the Boussinesq equation, and the KP equation with variable coefficient. Solitons are formed due to neglecting the nonlinearity and dispersion effect. Thus, they play an essential role in analyzing shallow water waves and occur in fields such as plasma physics, oceanography, marine engineering, fluid dynamics, dusty plasma, and other nonlinear sciences.

Keywords: Hirota direct method, Bilinear form, Cole-Hopf transformation, KdV equation, KP equation, Boussinesq equation, N-Soliton solutions.

1 Introduction

In physics and mathematics, a nonlinear partial differential equation (PDE) is an equation containing partial derivatives and nonlinear terms. PDEs depict many physical systems, from fluid dynamics to plasma physics and shallow water waves to oceanography, and have been used to solve different conjectures, such as the Poincaré conjecture and the Calabi conjecture. There is no general technique to solve nonlinear PDEs; therefore, we study every equation as an individual problem. Several techniques, such as Darboux transformation [1, 2], Bäcklund transformation [3, 4], Hirota bilinear technique [5–9], simplified Hirota method [10–13], Lie symmetry analysis [14, 15], Inverse scattering method [16, 17], Pfaffian technique [18, 19] and several other techniques are being used to study the nonlinear PDEs.

Among the several methods to study nonlinear

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PDEs, the Hirota direct method [5] is the most efficient tool for constructing multi-soliton solutions of an integrable nonlinear PDE. Hirota method provides a direct approach to get the exact solutions such as solitons, breathers, rogue waves, lumps, and others. In this method, the critical step is to convert the equation into a bilinear form suggested by Hirota and then apply the dependent variable transformation to obtain the multi-solitons of the nonlinear PDEs. As the concerned method is applied to integrable nonlinear PDE, confirming the integrability of a PDE plays an important role. Investigation of integrability to the nonlinear PDEs helps to generate multi-solitons as the integrable PDEs carry exponentially localized solutions. We use the Painlevé test [20] to check the complete integrability for a nonlinear PDE and use symbolic software such as *Mathematica/Maple* to perform such a tedious analysis. In 2006, Baldwin and Hereman [11] gave a symbolic computation for Painlevé analysis using *Mathematica* by applying the WTC-Krushkal method [21]. Several researchers and scientists have been attracted to construct soliton solutions derived from nonlinear PDEs due to their applicability in exhibiting practical features in nonlinear dynamics and ocean engineering dimensions.

This investigation explores Hirota's direct method on the KdV equation, which Hirota initially used to clarify his method. We apply the perturbation technique to get the mathematical formulation for soliton solutions as applied in this method. Also, we construct one soliton, two solitons, and three solitons and their generalized N-soliton solution. We show the bilinear form for the selected KdV equation and three well-known equations, the Kadomtsev-Petviashvili (KP) equation, the Boussinesq equation, and the KP equation with variable coefficient. Since solitons are formed due to ignoring the nonlinearity and dispersion effect; thus, they play an essential role in analyzing shallow water waves. Furthermore, occur in several fields such as plasma physics, oceanography, marine engineering, dusty plasma, fluid dynamics, and other sciences.

The structure of this work is as follows: Next section investigates the Hirota bilinear technique on the KdV equation. In Section 3, we apply Hirota's direct method to the different nonlinear PDEs, such as the Boussinesq equation, the Kadomtsev-Petviashvili equation, and the KP equation with variable coefficient, to construct the multiple solitons. In the last section, we conclude the work and investigation.

2 Hirota's Direct Method

To understand this method, we explore the steps of this technique to the integrable nonlinear Korteweg–De Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0; \qquad u = u(x,t). \tag{1}$$

We consider the solution for the equation (1) using Cole-Hopf transformation, which is a logarithmic transformation for the dependent variable u as

$$u = R(log f)_{xx}, \tag{2}$$

where f is a function of x and t and the coefficient R to be determined later. We can write the equation (2) in another form as

$$u = w_{xx} \qquad \text{where} \qquad w = R(log f). \tag{3}$$

Hirota [5] creates the bilinear form of the equation (1) by substituting,

$$u = e^{\alpha_i}, \tag{4}$$

into an equation with linear terms of Eq. (1) by considering the value of α_i (**phase variable**) as

$$\alpha_i = k_i x + w_i t, \tag{5}$$

where k_i and w_i are the constants (wave numbers) and the dispersions (frequencies), respectively. Thus, we get w_i in terms of k_i, known as **dispersion relation** which relates the wave numbers to their frequencies as

$$w_i = -k_i^3. \tag{6}$$

Next, we find the value of R in equation (2). For this, we consider the auxiliary function f in the Cole-Hopf transformation as

$$f(x,t) = 1+e^{\alpha_1} = 1+e^{k_1x+w_1t} = 1+e^{k_1x-k_1^3t}, \tag{7}$$

Furthermore, substitute in the equation (1). Then, on solving for R, we get R=2.
So, the logarithmic transformation becomes

$$u = 2(log f)_{xx}. \tag{8}$$

Now, from (3), we have

$$u_t = w_{xxt}, u_x = w_{xxx} \quad \text{and} \quad u_{xxx} = w_{xxxxx},$$

putting the above expressions into Eq. (1), we get

$$w_{xxt} + 6w_{xx}w_{xxx} + w_{xxxxx} = 0, \tag{9}$$

on integrating w.r.t. $'x'$

$$w_{xt} + 6\int w_{xx}w_{xxx}\partial x + w_{xxxx} = 0. \tag{10}$$

We compute integral term in equation (10) with constant of integration as zero

$$I = \int w_{xx}w_{xxx}\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\partial x = \frac{1}{2}w_{xx}^2, \tag{11}$$

substituting the value of I in equation (10), we get

$$w_{xt} + 3w_{xx}^2 + w_{xxxx} = 0. \tag{12}$$

As we have $w = 2(logf)$, we can get the followings:

$$w_x = 2\frac{f_x}{f}, \quad w_{xt} = 2\frac{ff_{xt} - f_xf_t}{f^2},$$

$$w_{xx} = \frac{ff_{xx} - f_x^2}{f^2},$$

$$w_{xxx} = 2\frac{2f_x^3 - 3ff_xf_{xx} + f^2f_{xxx}}{f^3},$$

$$w_{xxxx} = 2\frac{-6f_x^4 + 12ff_xf_{xx} - 3f^2f_{xx}^2 - 4f_xf_{xxx}}{f^4},$$

putting all the above values in (12), we get a quadratic equation in f as

$$-2\frac{f_xf_t}{f^2} + 2\frac{f_{xt}}{f} + 6\frac{f_{xx}^2}{f^2} - 8\frac{f_xf_{xxx}}{f^2} + 2\frac{f_{xxxx}}{f} = 0,$$

or

$$-f_xf_t + f_{xt} + 3f_{xx}^2 - 4f_xf_{xxx} + ff_{xxxx} = 0, \tag{13}$$

which can be written in terms of the operator D as

$$(D_xD_t + D_x^4)f.f = 0. \tag{14}$$

This equation (14) is called the **Hirota bilinear form** for equation (1) and the D-operator is defined as

$$D_x^lD_t^mg.h = (\partial_x - \partial_{x'})^l(\partial_t - \partial_{t'})^mg.h|_{x'=x,t'=t}, \tag{15}$$

where $g = g(x,t)$ and $h = h(x',t')$ are being differential functions and l and m are being non-negative integer.
As we have

$$u = 2(logf)_{xx} = \frac{2(ff_{xx} - f_x^2)}{f^2} = \frac{G}{F}, \tag{16}$$

where $G = 2(ff_{xx} - f_x^2)$ and $F = f^2$. The solution does not need to be the same if we take the solution of the equation (1) initially as $u = \frac{G}{F}$.
Solving equation (16) for f by putting $u = 0$, we get

$$f = e^{xC_1(t)}C_2(t).$$

On taking the constants $C_1(t) = 0$ and $C_2(t) = 1$, gives **zero-soliton solution** as

$$f_0 = 1. \tag{17}$$

Now we use the perturbation technique; we take f as a power series with a small parameter ϵ as

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots + \infty,$$

with Eq. (17)

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots + \epsilon^n f_n; \qquad n \to \infty. \tag{18}$$

By putting f from (18) in the equation (14), we have

$$(D_xD_t + D_x^4)(1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots)^2 = 0, \tag{19}$$

$$(D_xD_t + D_x^4)(1 + \epsilon(2f_1) + \epsilon^2(2f_2 + f_1f_1) + \cdots = 0. \tag{20}$$

Now collecting the terms of each order of ϵ equating to zero, we get

ϵ : $\quad (D_xD_t + D_x^4)f_1 = 0$

ϵ^2 : $\quad 2(D_xD_t + D_x^4)f_2 = -(D_xD_t + D_x^4)f_1.f_1$

ϵ^3 : $\quad 2(D_xD_t + D_x^4)f_3 = -2(D_xD_t + D_x^4)f_1.f_2$

$\vdots$

For D-operator, we have the following relations as

$$D_xD_tf.1 = f_{xt} = D_xD_t1.f, \tag{21}$$

$$D_x^4f.1 = f_{xxxx} = D_x^41.f, \tag{22}$$

$$D_x^mD_t^ne^{\theta_1}e^{\theta_2} = (k_1 - k_2)^m(w_1 - w_2)^ne^{\theta_1+\theta_2}, \tag{23}$$

where $\theta_i = k_ix + w_it + \theta_i^0$ with θ_i^0 (phase constant) $\to 0$.
Now for the coefficient of ϵ:

$$(D_xD_t + D_x^4)f_1 = 0,$$

$$D_xD_tf_1.1 + D_x^4f_1.1 = 0,$$

$$(f_1)_{xt} + (f_1)_{xxxx} = 0,$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}f_1 + \frac{\partial^4}{\partial x^4}f_1 = 0,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right)f_1 = 0,$$

which is a linear PDE for f_1. Thus the one-soliton solution is given by

$$f_1 = e^{\theta_1}, \tag{24}$$

where

$$\theta_1 = P_1 x + Q_1 t,$$

with

$$Q_1 = -P_1^3.$$

(Above expression for θ_1 is same as phase variable α_1 in equation (5) for $i = 1$ with condition as in equation (6).)
Again, for the coefficient of ϵ^2:

$$\begin{aligned} 2(D_x D_t + D_x^4) f_2 &= -(D_x D_t + D_x^4) f_1.f_1 \\ &= -(D_x D_t + D_x^4) e^{\theta_1}.e^{\theta_1} \\ &= 0 \end{aligned} \tag{25}$$

since we have $(D_x D_t + D_x^4)e^{\theta_1}.e^{\theta_1} = 0$ from Eq. (23).
Therefore, we can choose $f_2 = 0$, such that expansion of f in the equation (18) may be truncated to a finite sum as

$$f = 1 + \epsilon f_1. \tag{26}$$

Taking $\epsilon = 1$ (ϵ can be absorbed into the phase constant θ_1^0).

$$f = 1 + f_1 = 1 + e^{\theta_1}. \tag{27}$$

This is an exact solution of the bilinear form equation (14), which conforms **one-soliton solution** with the equation (8) as

$$\begin{aligned} u &= 2\left(\log(1+e^{\theta_1})\right)_{xx} = 2\left(\frac{e^{\theta_1}P_1}{1+e^{\theta_1}}\right)_x \\ &= 2\left(\frac{(1+e^{\theta_1})e^{\theta_1}P_1^2 - P_1 e^{\theta_1}.e^{\theta_1}P_1}{(1+e^{\theta_1})^2}\right) \\ &= \frac{2P_1^2 e^{\theta_1}}{(1+e^{\theta_1})^2} \end{aligned} \tag{28}$$

or

$$\begin{aligned} u &= \frac{2P_1^2 e^{\theta_1}}{1+e^{2\theta_1}+2e^{\theta_1}} = \frac{2P_1^2}{e^{-\theta_1}+e^{\theta_1}+2} \\ &= \frac{P_1^2}{\frac{e^{\theta_1}+e^{-\theta_1}}{2}+1} = \frac{P_1^2}{\cos\theta_1 + 1} \\ &= \frac{P_1^2}{2}\operatorname{sech}^2\left(\frac{\theta_1}{2}\right), \end{aligned} \tag{29}$$

where $\theta_1 = P_1 x + Q_1 t$ with $Q_1 = -P_1^3$; P_1 and Q_1 are arbitrary constants.

If we apply the principle of linear superposition for the solution f_1, we have

$$f_1 = e^{\theta_1} + e^{\theta_2}, \tag{30}$$

where $\theta_1 = P_1 x + Q_1 t$ and $\theta_2 = P_2 x + Q_2 t$ with dispersion relation as $Q_1 = -P_1^3$ and $Q_2 = -P_2^3$.
Now again, for the coefficient of ϵ^2:

$$\begin{aligned} 2(D_x D_t &+ D_x^4) f_2 = -(D_x D_t + D_x^4) f_1.f_1 \\ &= -(D_x D_t + D_x^4)(e^{\theta_1}+e^{\theta_2})^2 \\ &= -2(D_x D_t + D_x^4).e^{\theta_1}.e^{\theta_2} \\ &= -2(D_x D_t e^{\theta_1}.e^{\theta_2} + D_x^4 e^{\theta_1}.e^{\theta_2} \\ &= -2((P_1-P_2)(Q_1-Q_2)e^{\theta_1}.e^{\theta_2} \\ &+ (P_1-P_2)^4 e^{\theta_1}.e^{\theta_2}), \end{aligned} \tag{31}$$

using (23), we get

$$= -2(P_1-P_2)\{(Q_1-Q_2)+(P_1-P_2)^3\}e^{\theta_1+\theta_2}. \tag{32}$$

We may consider the solution of Eq. (32) as

$$f_2 = a_{12} e^{\theta_1+\theta_2} \tag{33}$$

where the coefficient a_{12} using (23) is

$$\begin{aligned} a_{12} &= \frac{-2(P_1-P_2)\{(Q_1-Q_2)+(P_1-P_2)^3\}}{2(P_1+P_2)\{(Q_1+Q_2)+(P_1+P_2)^3\}} \\ &= \frac{-(P_1-P_2)\{(-P_1^3+P_2^3)+(P_1-P_2)^3\}}{(P_1+P_2)\{(-P_1^3-P_2^3)+(P_1+P_2)^3\}} \\ &= \frac{-(P_1-P_2)\{-3P_1^2+3P_1P_2^2\}}{(P_1+P_2)\{3P_1^2+3P_1P_2^2\}} \\ &= \frac{(P_1-P_2)^2}{(P_1+P_2)^2}. \end{aligned} \tag{34}$$

Substituting f_1 and f_2 in equation of coefficient for ϵ_3

$$\begin{aligned} 2(D_x D_t &+ D_x^4) f_3 = -2(D_x D_t + D_x^4) f_1.f_2 \\ &= -2a_{12}(D_x D_t + D_x^4)(e^{\theta_1}+e^{\theta_2}).e^{\theta_1+\theta_2} \\ &= -2a_{12}(D_x D_t + D_x^4)e^{2\theta_1+\theta_2}.e^{\theta_1+2\theta_2} \\ &= 0. \end{aligned} \tag{35}$$

So, we may choose $f_3 = 0$; therefore, the expansion of f in the equation (18) may be truncated to a finite sum as

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 = 1 + \epsilon(e^{\theta_1}+e^{\theta_2}) + \epsilon^2 a_{12} e^{\theta_1+\theta_2}. \tag{36}$$

Thus, for $\epsilon = 1$, we have

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}, \tag{37}$$

which is an exact solution of Eq. (14) that conforms a **two-soliton solution** by the Eq. (8).
Similarly, we can find a three-soliton solution by choosing

$$f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{38}$$

where $\theta_i = P_i x + Q_i t$ for $i = 1, 2, 3$ with dispersion relation as $Q_i = -P_i^3$.
So, we get the expansion of f in the equation (18) for $\epsilon = 1$ as

$$f = 1 + f_1 + f_2 + f_3$$
$$= 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{1=i<j}^{3} a_{ij} e^{\theta_i+\theta_j} + a_{123} e^{\theta_1+\theta_2+\theta_3}, \tag{39}$$

where

$$a_{123} = a_{12} a_{13} a_{23}, \tag{40}$$

and

$$a_{ij} = \frac{(P_i - P_j)^2}{(P_i + P_j)^2}. \tag{41}$$

By the function f in Eq. (39), we get the **three-soliton solution** with Eq. (8).
Hence, by repeating the same procedure, we can create a closed expression for the auxiliary function f to get the **N-soliton solution** as

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{1=i<j}^{N} A_{ij} \mu_i \mu_j \right), \tag{42}$$

where $\sum_{\mu=0,1}$ indicates the summation of all possible combinations for $\mu_i = 0, 1$ for $1 \leq i \leq N$.
For $N = 1$, we have $\mu_1 = 0, 1$ so $f = 1 + e^{\eta_1}$
For $N = 2$, we have $\mu_1 = 0, 1$ and $\mu_2 = 0, 1$. So there will be four combinations of μ_1 and μ_2 as $(0,0), (0,1), (1,0)$ and $(1,1)$, thus the function f will be as

$$f = 1+e^{\eta_1}+e^{\eta_2}+e^{A_{12}+\eta_1+\eta_2} = 1+e^{\eta_1}+e^{\eta_2}+a_{12}e^{\eta_1+\eta_2},$$

with $a_{12} = e^{A_{12}}$.
For $N = 3$, we have $\mu_1, \mu_2, \mu_3 = 0, 1$ so the total combinations for μ_1, μ_2, and μ_3 will be eight as {(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)}, thus the expression for f will be as

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{A_{12}+\eta_1+\eta_2} + e^{A_{13}+\eta_1+\eta_3}$$
$$+ e^{A_{23}+\eta_2+\eta_3} + e^{A_{12}+A_{13}+A_{23}+\eta_1+\eta_2+\eta_3},$$

or

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + a_{12}e^{\eta_1+\eta_2} + a_{13}e^{\eta_1+\eta_3}$$
$$+ a_{23}e^{\eta_2+\eta_3} + a_{123}e^{\eta_1+\eta_2+\eta_3},$$

with $a_{ij} = e^{A_{ij}}$ and $a_{123} = e^{A_{12}+A_{13}+A_{23}}$.

3 Application of Hirota method to the nonlinear PDEs

The above section discusses the procedure for Hirota's direct method with different steps to get N-soliton solutions. We can summarise the steps as

- Considering the phase variable θ_i depending on the given nonlinear PDE
- Finding dispersion relation, a relation between frequencies and wave numbers.
- Finding Cole-Hopf transformation $u = R(logf)_{x^n}$ for the given nonlinear PDE, where n is the order of partial differentiation w.r.t. x depending upon the balance of higher order term and nonlinear term in the PDE.
- Finding the Bilinear form of the nonlinear PDE.
- Apply the N-soliton solutions formulation for the auxiliary function f to the bilinear form to get the values of the constants appearing in f.
  - For $N = 1$, we take $f = 1 + e^{\theta_1}$ where θ_1 is the phase variable.
  - For $N = 2$, we get $f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$ where $\theta_i; i = 1, 2$ are phase variables and a_{12} is constant.
  - For $N = 3$, we have $f = 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{1=i<j}^{3} a_{ij}e^{\theta_i+\theta_j} + a_{123}e^{\theta_1+\theta_2+\theta_3}$ where $\theta_i; i = 1, 2, 3$ are phase variables and $a_{ij}; 1 \leq i < j \leq 3$, and a_{123} are constants.
- Computing soliton solutions as $u = R(logf)_{x^n}$ concerning the choice of f as above for different N.

3.1 (2+1)-dimensional Kadomtsev-Petviashvili equation

We have the integrable KP equation [22, 23] as

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0 \qquad (43)$$

- We consider the phase variable $\theta_i = p_i x + q_i y - w_i t$
- We get the dispersion relation by putting $u = e^{\theta_i}$ in the linear terms of the Eq. (43) as $w_i = \frac{p_i^4 + q_i^2}{p_i}$.
- We consider the solutions as Cole-Hopf transformation $u = R(logf)_{xx}$, where we get $R = 2$ by putting the function $f = 1 + e^{\theta_1}$ into the equation (43).
- We create the Bilinear form of the equation as $(D_x^4 + D_x D_t + D_y^2)f.f = 0$.
- We obtain directly the soliton solutions for the function f as
  - For $N = 1$, we take $f = 1 + e^{\theta_1}$ where $\theta_1 = p_1 x + q_1 y - \left(\frac{p_1^4 + q_1^2}{p_1}\right) t$, so we get one-soliton solution as $u = 2(log(1 + e^{\theta_1}))_{xx} = \frac{2k_1^2 e^{\theta_1}}{\left(1+e^{\theta_1}\right)^2}$
  - For $N = 2$, we have $f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$ where $\theta_i = p_i x + q_i y - (\frac{p_i^4 + q_i^2}{p_i})t; i = 1, 2$ and compute the constant as $a_{12} = \frac{3p_1^2 p_2^2 (p_1 - p_2)^2 - (p_1 q_2 - p_2 q_1)^2}{3p_1^2 p_2^2 (p_1 + p_2)^2 - (p_1 q_2 - p_2 q_1)^2}$.Thus we get a two-soliton solution as $u = 2(log(1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}))_{xx}$.
  - For $N = 3$, we have $f = 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{1=i<j}^{3} a_{ij}e^{\theta_i+\theta_j} + a_{123}e^{\theta_1+\theta_2+\theta_3}$ where $\theta_i = p_i x + q_i y - (\frac{p_i^4 + q_i^2}{p_i})t; i = 1, 2, 3$ and $a_{123} = a_{12}a_{13}a_{23}$ with $a_{ij} = \frac{3p_i^2 p_j^2 (p_i - p_j)^2 - (p_i q_j - p_j q_i)^2}{3p_i^2 p_j^2 (p_i + p_j)^2 - (p_i q_j - p_j q_i)^2}$; $1 \le i < j \le 3$. Thus we get a three-soliton solution by $u = 2(logf)_{xx}$.

3.2 (1+1)-dimensional Boussinesq equation

We have the integrable Boussinesq equation [24] as

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0 \qquad (44)$$

- We consider the phase variable $\theta_i = p_i x - w_i t$
- We get the dispersion relation by putting $u = e^{\theta_i}$ in the linear terms of the equation (44) as $w_i = -\sqrt{p_i^2 + p_i^4}$.
- We consider the solutions as Cole-Hopf transformation $u = R(logf)_{xx}$, where we get $R = 2$ by putting the function $f = 1 + e^{\theta_1}$ into the equation (44).
- We create the Bilinear form of the Eq. (44) as $(D_t^2 - D_x^2 - D_x^4)f.f = 0$.
- We get directly the soliton solutions for the function f as
  - For $N = 1$, we take $f = 1 + e^{\theta_1}$ where $\theta_1 = p_1 x + \left(\sqrt{p_1^2 + p_1^4}\right) t$, so we get one-soliton solution as $u = 2(log(1 + e^{\theta_1}))_{xx} = \frac{2p_1^2 e^{\theta_1}}{\left(1+e^{\theta_1}\right)^2}$.
  - For $N = 2$, we have $f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$ where $\theta_i = p_i x + \left(\sqrt{p_i^2 + p_i^4}\right) t; i = 1, 2$ and compute the constant as $a_{12} = \frac{\sqrt{1+p_1^2}\sqrt{1+p_2^2} - (2p_1^2 - 3p_1 p_2 + 2p_2^2 + 1)}{\sqrt{1+p_1^2}\sqrt{1+p_2^2} - (2p_1^2 + 3p_1 p_2 + 2p_2^2 + 1)}$.Thus we get a two-soliton solution as $u = 2(log(1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}))_{xx}$.
  - For $N = 3$, we have $f = 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{1=i<j}^{3} a_{ij}e^{\theta_i+\theta_j} + a_{123}e^{\theta_1+\theta_2+\theta_3}$ where $\theta_i = p_i x + \left(\sqrt{p_i^2 + p_i^4}\right) t; i = 1, 2, 3$ and $a_{123} = a_{12}a_{13}a_{23}$ with $a_{ij} = \frac{\sqrt{1+p_i^2}\sqrt{1+p_j^2} - (2p_i^2 - 3p_i p_j + 2p_j^2 + 1)}{\sqrt{1+p_i^2}\sqrt{1+p_j^2} - (2p_i^2 + 3p_i p_j + 2p_j^2 + 1)}$; $1 \le i < j \le 3$. Thus we get a three-soliton solution by $u = 2(logf)_{xx}$.

3.3 (2+1)-dimensional KP equation with variable coefficient

Integrable KP equation with variable coefficient [25, 26] is given by

$$(u_t + uu_x + u_{xxx})_x + 3u_{yy} + g(t)u_{xy} = 0, \qquad (45)$$

- We consider the phase variable $\theta_i = p_i x + q_i y - w_i(t)$
- We get the dispersion relation by putting $u = e^{\theta_i}$ in the linear terms of the equation (45) as $w_i = \int \left(p_i^3 + g(t)q_i + \frac{3q_i^2}{p_i}\right) dt$.

- We consider the solutions as Cole-Hopf transformation $u = R(logf)_{xx}$, where we get $R = 12$ by putting the function $f = 1 + e^{\theta_1}$ into the equation (45).
- We create the Bilinear form of Eq. (45) as $(D_x^4 + g(t)D_xD_y + D_xD_t + 3D_y^2)f.f = 0$.
- We get directly the soliton solutions for the function f as
  - For $N = 1$, we have $f = 1 + e^{\theta_1}$ where $\theta_1 = p_1x + q_1y - w_1(t)$, so we get one-soliton solution as $u = 12p_1^2 \frac{e^{p_1x+q_1y-w_1(t)}}{(e^{w_1(t)}+e^{p_1x+q_1y})^2}$.
  - For $N = 2$, we take $f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$ where $\theta_i = p_ix + q_iy - w_i(t); i = 1, 2$ and compute the constant as $a_{12} = \frac{p_2^2(p_1^2(p_1-p_2)^2-q_1^2)+2p_1p_2q_1q_2-p_1^2p_2^2}{p_2^2(p_1^2(p_1+p_2)^2-q_1^2)+2p_1p_2q_1q_2-p_1^2p_2^2}$.Thus we get a two-soliton solution as $u = 12(log(1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}))_{xx}$.
  - For $N = 3$, we have $f = 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{1=i<j}^{3} a_{ij}e^{\theta_i+\theta_j} + a_{123}e^{\theta_1+\theta_2+\theta_3}$ where $\theta_i = p_ix + q_iy - w_i(t); i = 1, 2, 3$ and $a_{123} = a_{12}a_{13}a_{23}$ with $a_{ij} = \frac{p_j^2(p_i^2(p_i-p_j)^2-q_i^2)+2p_ip_jq_iq_j-p_i^2p_j^2}{p_j^2(p_i^2(p_i+p_j)^2-q_i^2)+2p_ip_jq_iq_j-p_i^2p_j^2}$; $1 \le i < j \le 3$. Thus we get a three-soliton solution by $u = 12(logf)_{xx}$.

4 Conclusions

In this work, we investigated Hirota's direct method on the KdV equation using the perturbation technique to get the N-soliton solution. We showed the perturbation technique to the KdV equation to generate one-soliton, two-soliton, and three-soliton solutions. We explored the closed expression of the N-soliton solution for the same. We created the bilinear form for the KdV equation and the other equations used in this work. Investigation of the concerned method has illustrated three well-known equations, the Kadomtsev-Petviashvili equation, the Boussinesq equation and the KP equation with variable coefficient. Solitons are formed due to neglecting the nonlinearity and dispersion effect. Thus, they play an essential role in analyzing shallow water waves and occur in fields such as plasma physics, oceanography, dusty plasma, marine engineering, fluid dynamics, and other nonlinear sciences.

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Conflict of interest

The authors declare that there is no conflict of interest.

Data availability statement

No data from outside sources has been used in this work.

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